

Integral representation of Skorokhod reflection

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Abstract

We show that a certain integral representation of the one-sided Skorokhod reflection of a continuous bounded variation function characterizes the reflection in that it possesses a unique maximal solution which solves the Skorokhod reflection problem.

1 Introduction

The Skorokhod reflection problem has a long history. Skorokhod [10] introduced it as a method for representing a diffusion process with a reflecting boundary at zero. Given a continuous function $X : [0, \infty) \rightarrow \mathbb{R}$, the standard Skorokhod reflection problem seeks to find $(Q(t), t \geq 0)$ and a continuous, nondecreasing function $Y : [0, \infty) \rightarrow \mathbb{R}_+$ with $Y(0) = 0$, such that $Q(t) := X(t) + Y(t) \geq 0$ for all t , and $\int_0^\infty Q(s) dY(s) = 0$. Intuitively, the latter expresses the idea that Y can increase only at points t such that $X(t) + Y(t) = 0$. Skorokhod [10] showed that there is only one such Y , namely, $Y(t) = -\inf_{0 \leq s \leq t} (X(s) \wedge 0)$ and thus

$$Q(t) = X(t) \vee \sup_{0 \leq s \leq t} (X(t) - X(s)).$$

We use the standard notation $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$. The mapping $X \mapsto Q$ is referred to as the (one-sided) Skorokhod reflection mapping and has now become a standard tool in probability theory and other areas. As an example, we recall that if X is the path of a Brownian motion then Q is a reflecting Brownian motion and $Q(t)$ has the same distribution as $|X(t)|$ for all $t \geq 0$ [3, 9]. Several extensions of the Skorokhod reflection mapping exist generalizing the range of X (see, e.g., [11]) or its domain (see, e.g., [1]).

The question resolved in this paper was motivated by an application of the Skorokhod reflection in stochastic fluid queues [7, 6]. Suppose that A, C are two jointly stationary and ergodic random measures defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with intensities a, c , respectively, such that $a < c$. Then there exists a unique stationary and ergodic

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stochastic process $(Q(t), t \in \mathbb{R})$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for all $t_0 \in \mathbb{R}$, $(Q(t_0 + t), t \geq 0)$ is the Skorokhod reflection of $(Q(t_0) + A(t_0, t_0 + t] - C(t_0, t_0 + t], t \geq 0)$. In addition, if the random measures A, C have no atoms then

$$Q(t) = \int_{-\infty}^t \mathbf{1}(Q(s) > C(s, t]) \, dA(s), \quad (1)$$

for all $t \in \mathbb{R}$, \mathbb{P} -almost surely. The latter equation was called an “integral representation” of Skorokhod reflection and extensions of it were formulated and proved in [6]. The integral representation was found to be useful in several applications, e.g. (i) in deriving the so-called Little’s law for stochastic fluid queues [2], stating that $\mathbb{E}[Q(0)] = (a/c)\mathbb{E}_A[Q(0)]$, where \mathbb{E}_A is expectation with respect to the Palm measure [4] of \mathbb{P} with respect to A , and (ii) in deriving the form of the stationary distribution of a stochastic process derived from the local time of a Lévy process [5].

In an open problems session of the workshop on “New Topics at the Interface Between Probability and Communications” [8], the second author asked whether and in what sense (1) characterizes Skorokhod reflection. The question will be made precise in Section 2 below, where the main theorem, Theorem 1, which answers the question, is stated. In Section 3 the integral representation is explicitly proved, along with some auxiliary results. Finally, in Section 4 a proof of Theorem 1 is given.

2 The problem

Consider a locally finite signed measure X on the Borel sets of \mathbb{R} . Assume that X has no atoms, i.e. $X(\{t\}) = 0$ for all $t \in \mathbb{R}$. Define

$$Q^*(t) := \sup_{0 \leq s \leq t} X(s, t], \quad t \geq 0, \quad (2)$$

where $X(s, t] = X((s, t])$ is the value of X at the interval $(s, t]$.¹ In particular,

$$Q^*(0) = 0.$$

Let $X(t) := X(0, t]$ and write (2) as

$$Q^*(t) = X(t) - \inf_{0 \leq s \leq t} X(s).$$

The standard terminology [3, 12] is that Q^* solves the Skorokhod reflection problem for the function $t \mapsto X(t)$.

Decompose X as the difference of two locally finite nonnegative measures A, C , without atoms, i.e. write

$$X = A - C. \quad (3)$$

We stress that A, C are not necessarily the positive and negative parts of X . In other words, the decomposition is not unique. For instance, we can add an arbitrary locally finite nonnegative measure without atoms to both A and C .

¹Since X, A, C are assumed to have no atoms, we may as well write $X[s, t]$ or $X(s, t)$ instead of $X(s, t]$, and likewise for A and C , but we have chosen the notation to be consistent with possible generalizations.

In [6] it was proved that (2) also satisfies the fixed point equation referred to as “integral representation” of the reflected process:

$$Q(t) = \int_0^t \mathbf{1}(Q(s) > C(s, t]) \, dA(s), \quad t \geq 0. \quad (4)$$

A simpler version of this appeared earlier in [7]; this version was concerned with the case where C is a multiple of the Lebesgue measure. In an open problems session of the workshop on “New Topics at the Interface Between Probability and Communications” [8], the second author asked whether and in what sense (4) implies (2); the question was actually asked for the special case where C is a multiple of the Lebesgue measure.

In this note we answer this question by proving the following:

Theorem 1. *Let A, C be locally finite Borel measures on $\mathbb{R}_+ = [0, \infty)$ without atoms and consider the integral equation (4). This integral equation admits a unique maximal solution, i.e. a solution which pointwise dominates any other solution. Further, this maximal solution is precisely the function Q^* defined by (2).*

We proceed as follows. First, we present some auxiliary results and also give a proof of (2) \Rightarrow (4) which is different from the one found in [6]. Then we prove Theorem 1 by a successive approximation scheme and by proving a number of lemmas.

3 Proof of the integral representation and auxiliary results

We first exhibit some properties of Q^* , defined by (2), and also show that Q^* satisfies the integral equation (4). The proof of the latter in the special case where C is a multiple of the Lebesgue measure can be found in [7, Lemma 1] and in [2, §3.5.3]. A more general case is dealt with in [6, Theorem 1]. We give a different proof in Proposition 1 below. The lemmas below are straightforward and well-known but we give proofs for completeness. As before, X is a locally finite Borel measure without atoms and $X = A - C$ is a decomposition as the difference of two nonnegative locally finite Borel measures without atoms. We set

$$A(t) := A(0, t], \quad C(t) := C(0, t].$$

Lemma 1. *If $0 \leq s \leq s' \leq t$ and if $Q^*(s) > C(s, t]$ then $Q^*(s') > C(s', t]$.*

Proof. Assume that $C(s, t] < Q^*(s) = \sup_{0 \leq u \leq s} X(u, s]$. This is equivalent to

$$\begin{aligned} C(t) - C(s) &< \sup_{0 \leq u \leq s} \{A(s) - A(u) - (C(s) - C(u))\} \\ &= A(s) + \sup_{0 \leq u \leq s} \{-A(u) + C(u)\} - C(s), \\ \text{that is, } C(t) &< A(s) + \sup_{0 \leq u \leq s} \{-A(u) + C(u)\}. \end{aligned}$$

The right-hand side of the latter is increasing in s and so replacing s by a larger s' we obtain

$$C(t) < A(s') + \sup_{0 \leq u \leq s'} \{-A(u) + cu\},$$

which is equivalent to $Q^*(s') > C(s', t]$. □

Lemma 2. Q^* satisfies

$$Q^*(t) = \sup_{s \leq u \leq t} X(u, t] \vee (Q^*(s) + X(s, t]), \quad 0 \leq s \leq t. \quad (5)$$

Proof. We show that the right-hand side of (5) equals the left-hand side.

$$\begin{aligned} \sup_{s \leq u \leq t} X(u, t] \vee (Q^*(s) + X(s, t]) &= \sup_{s \leq u \leq t} X(u, t] \vee \{(\sup_{0 \leq u \leq s} X(u, s]) + X(s, t])\} \\ &= \sup_{s \leq u \leq t} X(u, t] \vee \sup_{0 \leq u \leq s} \{X(u, s] + X(s, t])\} \\ &= \sup_{s \leq u \leq t} X(u, t] \vee \sup_{0 \leq u \leq s} X(u, t] \\ &= \sup_{0 \leq u \leq t} X(u, t] = Q^*(t). \end{aligned}$$

Lemma 3. If $0 \leq s \leq t$ and if $Q^*(s) \geq C(s, t]$ then $Q^*(t) = Q^*(s) + X(s, t]$.

Proof. We use equation (5), rewritten as follows:

$$Q^*(t) = \sup_{s \leq u \leq t} \{X(u, t] \vee (Q^*(s) + X(s, t])\}. \quad (6)$$

Suppose $0 \leq s \leq u \leq t$ and that $Q^*(s) \geq C(s, t]$. Then $Q^*(s) \geq C(s, u]$ and so

$$\begin{aligned} Q^*(s) + X(s, t] &\geq C(s, u] + X(s, t] \\ &= C(s, u] + A(s, t] - C(s, t] \\ &= A(s, t] - C(u, t] \\ &\geq A(u, t] - C(u, t] = X(u, t], \end{aligned}$$

and this inequality implies that the term $X(u, t]$ inside the bracket of the right-hand side of (6) is not needed. Hence $Q^*(t) = Q^*(s) + X(s, t]$, which is what we wanted to prove. \square

Define next

$$\sigma^*(t) := \sup\{0 \leq s \leq t : Q^*(s) \leq C(s, t]\}.$$

By Lemma 1,

$$Q^*(s) \leq C(s, t], \quad \text{if } 0 \leq s \leq \sigma^*(t), \quad (7a)$$

$$Q^*(s) > C(s, t], \quad \text{if } \sigma^*(t) < s \leq t, \quad (7b)$$

provided that the last inequality is non-vacuous. Since the function Q^* is nonnegative and continuous, we also have

$$Q^*(\sigma^*(t)) = C(\sigma^*(t), t].$$

Proposition 1. If X is a locally finite signed Borel measure on $[0, \infty)$ without atoms and if $X = A - C$ is any decomposition of X as the difference of two nonnegative locally finite Borel measures without atoms, then the function Q^* defined by (2) satisfies (4).

Proof. By Lemma 3, and the last display,

$$\begin{aligned}
Q^*(t) &= Q^*(\sigma^*(t)) + A(\sigma^*(t), t] - C(\sigma^*(t), t] \\
&= A(\sigma^*(t), t] \\
&= \int_{\sigma^*(t)}^t dA(s) \\
&= \int_0^t \mathbf{1}(Q^*(s) > C(s, t]) \, dA(s),
\end{aligned}$$

which is the integral representation formula (4). Note that, to obtain the last equality in the last display, we used (7a)-(7b). \square

4 Proof of Theorem 1

A priori, it is not clear that (4) admits a maximal solution and, even if it does, whether it satisfies (2). We shall show the validity of these claims in the sequel.

We fix two locally finite measures A and C and define the map Θ on the set of nonnegative measurable functions by

$$\Theta(Q)(t) := \int_0^t \mathbf{1}(Q(s) > C(s, t]) \, dA(s), \quad t \geq 0. \quad (8)$$

The integral equation (4) then reads

$$Q = \Theta(Q).$$

We observe that Θ is increasing:

$$\text{If } Q \leq \tilde{Q} \text{ then } \Theta(Q) \leq \Theta(\tilde{Q}). \quad (9)$$

Here, and in the sequel, given two functions $f, g : [0, \infty) \rightarrow \mathbb{R}$, we write $f \leq g$ to mean that $f(t) \leq g(t)$ for all $t \geq 0$. To see that (8) holds, simply observe that $Q \leq \tilde{Q}$ implies $\mathbf{1}(Q(s) > C(s, t]) \leq \mathbf{1}(\tilde{Q}(s) > C(s, t])$ for all $0 \leq s \leq t$.

Define next a sequence of functions $(Q_k, k = 0, 1, 2, \dots)$ by first letting

$$Q_0 := \infty,$$

and then, recursively,

$$Q_{k+1} := \Theta(Q_k), \quad k \geq 0.$$

Clearly, $Q_1(t) = \int_0^t dA(s) = A(t)$. So $Q_0 \geq Q_1$. Since Θ is an increasing map, we see that,

$$Q_k \geq Q_{k+1} \geq 0, \quad k \geq 0.$$

We can then define

$$Q_\infty(t) := \lim_{k \rightarrow \infty} Q_k(t).$$

Lemma 4. *If $Q = \Theta(Q)$ then $Q \leq Q_\infty$. Furthermore,*

$$Q^* \leq Q_\infty.$$

Proof. Suppose that Q satisfies $Q = \Theta(Q)$. Since the integrand in the right-hand side of (8) is ≤ 1 , we have $Q(t) \leq A(t)$ for all $t \geq 0$. Letting $\Theta^{(k)}$ be the k -fold composition of Θ with itself, we have

$$Q = \Theta^{(k)}(Q) \leq \Theta^{(k)}(A) = Q_k,$$

and so $Q \leq Q_\infty$. In particular, Proposition 1 states that $Q^* = \Theta(Q^*)$. Hence $Q^* \leq Q_\infty$. \square

However, it is not yet clear at this point that Q_∞ is a fixed point of Θ . We can only show that

$$Q_\infty \geq \Theta(Q_\infty).$$

Indeed, $Q_\infty \leq Q_k$ for all k , and so $\mathbf{1}(Q_\infty(s) > C(s, t]) \leq \mathbf{1}(Q_k(s) > C(s, t])$, for all $0 \leq s \leq t$, implying that $\Theta(Q_\infty) \leq \Theta(Q_k) = Q_{k+1}$, and, by taking limits, that $\Theta(Q_\infty) \leq Q_\infty$.

Definition 1 (Regulating functions). *Consider functions $B : [0, \infty) \rightarrow [0, \infty)$ which are continuous, nondecreasing, with $B(0) = 0$, such that $X(0, t] + B(t) \geq 0$ for all $t \geq 0$. Call these functions regulating functions of X . The set of regulating functions is denoted by $\mathcal{R}(X)$.*

We define a mapping

$$\Phi : \mathcal{R}(X) \rightarrow \mathcal{R}(X) \tag{10}$$

in two steps: Given $B \in \mathcal{R}(X)$, first define

$$\sigma_B(t) := \sup\{0 \leq s \leq t : A(s) + B(s) - C(t) \leq 0\}, \quad t \geq 0.$$

Then let

$$\Phi(B)(t) := B(\sigma_B(t)), \quad t \geq 0.$$

We actually need to show that what is claimed in (10) holds. Namely:

Lemma 5. *If $B \in \mathcal{R}(X)$ then $\Phi(B) \in \mathcal{R}(X)$.*

Proof. Clearly, $\sigma_B(\cdot)$ is nondecreasing. Since B is nondecreasing, it follows that $\Phi(B) = B \circ \sigma_B$ is nondecreasing. Also, $\Phi(B)(0) = B(\sigma_B(0)) = B(0) = 0$. From the continuity of A , B and the definition of σ_B , we have

$$A(\sigma_B(t)) + B(\sigma_B(t)) = C(t), \quad t \geq 0. \tag{11}$$

We also have,

$$\begin{aligned} A(t) + \Phi(B)(t) - C(t) &= A(t) + B(\sigma_B(t)) - C(t) \\ &= [A(t) - A(\sigma_B(t))] + [A(\sigma_B(t)) + B(\sigma_B(t)) - C(t)] \\ &= A(t) - A(\sigma_B(t)) \geq 0, \end{aligned}$$

where we used (11) in the third step. It remains to show that $\Phi(B)(\cdot)$ is continuous. Note that $\sigma_B(\cdot)$ need not be continuous. However, $C(\cdot)$ is a continuous function and so, by (11), $t \mapsto A(\sigma_B(t)) + B(\sigma_B(t))$ is continuous. Hence

$$[A(\sigma_B(t+)) - A(\sigma_B(t-))] + [B(\sigma_B(t+)) - B(\sigma_B(t-))] = 0, \quad \text{for all } t.$$

Since $A(\sigma_B(\cdot))$ and $B(\sigma_B(\cdot))$ are both nondecreasing, it follows that $A(\sigma_B(t+)) - A(\sigma_B(t-)) \geq 0$ and $B(\sigma_B(t+)) - B(\sigma_B(t-)) \geq 0$ and, since their sum is zero, they are both zero, implying that $A(\sigma_B(\cdot))$ and $B(\sigma_B(\cdot))$ are continuous. \square

An immediate property of Φ is that

$$\Phi(B) \leq B \quad \text{for all } B \in \mathcal{R}(X). \quad (12)$$

Indeed, for all $t \geq 0$, $\sigma_B(t) \leq t$ and so $B(\sigma_B(t)) \leq B(t)$.

Starting with the function

$$B_1(t) := C(t), \quad t \geq 0, \quad (13)$$

we recursively define

$$B_{k+1} := \Phi(B_k), \quad k \geq 1. \quad (14)$$

Therefore

$$B_1 \geq B_2 \geq \cdots \geq B_k \downarrow B_\infty, \quad \text{as } k \rightarrow \infty, \quad (15)$$

where the inequalities and the limit are pointwise.

Lemma 6. *The function B_∞ , defined via (13), (14) and (15), is a member of the class $\mathcal{R}(X)$.*

Proof. B_∞ is nondecreasing since all the B_k are nondecreasing. Also, $B_\infty(0) = 0$. Since for all k , $A + B_k - C \geq 0$, we have $A + B_\infty - C \geq 0$. We proceed to show that B_∞ is a continuous function. We observe that, for $0 \leq t \leq t'$,

$$\begin{aligned} |\Phi(B)(t') - \Phi(B)(t)| &= |B(\sigma_B(t')) - B(\sigma_B(t))| \\ &= B(\sigma_B(t')) - B(\sigma_B(t)) \\ &\leq A(\sigma_B(t')) - A(\sigma_B(t)) + B(\sigma_B(t')) - B(\sigma_B(t)) \\ &= [A(\sigma_B(t')) + B(\sigma_B(t'))] - [A(\sigma_B(t)) + B(\sigma_B(t))] \\ &= C(t') - C(t), \end{aligned}$$

where we again used (11). It follows that the family of functions $\{\Phi(B), B \in \mathcal{R}(X)\}$ is uniformly bounded and equicontinuous on each compact interval of the real line. By the Arzelà-Ascoli theorem, the family is compact and therefore B_∞ is continuous. We have established that $B_\infty \in \mathcal{R}(X)$. \square

We now claim that B_∞ is a fixed point of Φ .

Lemma 7. $\Phi(B_\infty) = B_\infty$.

Proof. By definition,

$$\Phi(B_\infty)(t) = B_\infty(\sigma_{B_\infty}(t)),$$

where

$$\sigma_{B_\infty}(t) = \sup\{0 \leq s \leq t : A(s) + B_\infty(s) \leq C(t)\}.$$

Now, since $B_k \geq B_{k+1}$ for all $k \geq 1$, it follows that $\sigma_{B_k} \leq \sigma_{B_{k+1}}$ for all $k \geq 1$, and so

$$\sigma_L(t) := \lim_{k \rightarrow \infty} \sigma_{B_k}(t)$$

is well-defined. Since $B_k \geq B_\infty$ for all $k \geq 1$, we have $\sigma_{B_k} \leq \sigma_{B_\infty}$. Taking limits, we find

$$\sigma_L \leq \sigma_{B_\infty}.$$

Using the last two displays and the fact that B_k and B_∞ are nondecreasing, we have

$$\begin{aligned} \Phi(B_\infty)(t) &= B_\infty(\sigma_{B_\infty}(t)) \geq B_\infty(\sigma_L(t)) \\ &= \lim_{k \rightarrow \infty} B_k(\sigma_L(t)) \\ &\geq \lim_{k \rightarrow \infty} B_k(\sigma_{B_k}(t)) \\ &= \lim_{k \rightarrow \infty} B_{k+1}(t) = B_\infty(t). \end{aligned}$$

By inequality (12), $\Phi(B) \leq B$ for all $B \in \mathcal{R}(X)$ and since, by Lemma 6, $B_\infty \in \mathcal{R}(X)$, it follows that we also have $B_\infty \leq \Phi(B_\infty)$. Therefore $B_\infty = \Phi(B_\infty)$, as claimed. \square

Lemma 8. Consider the function Q^* defined by (2) and define a function U by

$$U(t) := Q^*(t) - X(0, t], \quad t \geq 0.$$

Then

$$(i) \ U \in \mathcal{R}(X).$$

$$(ii) \ U = \Phi(U).$$

Proof. (i) We have $X(0, t] + U(t) = Q^*(t) \geq 0$ for all t . Using (2) and (3) we see that

$$U(t) = \sup_{0 \leq s \leq t} \{-A(s) + C(s)\}. \quad (16)$$

Therefore, $U(0) = 0$, and U is a continuous and nondecreasing. We conclude that $U \in \mathcal{R}(X)$.

To prove (ii), recall that $\Phi(U) = U \circ \sigma_U$ where

$$\sigma_U(t) = \sup\{0 \leq s \leq t : A(s) + U(s) \leq C(t)\}.$$

Splitting the supremum in (16) in two parts, we obtain

$$\begin{aligned} U(t) &= \sup_{0 \leq s \leq \sigma_U(t)} \{-A(s) + C(s)\} \vee \sup_{\sigma_U(t) \leq s \leq t} \{-A(s) + C(s)\}. \\ &= U(\sigma_U(t)) \vee \sup_{\sigma_U(t) \leq s \leq t} \{-A(s) + C(s)\}. \end{aligned}$$

For $s \geq \sigma_U(t)$, we have $A(s) + U(s) \geq C(t)$, i.e. $-A(s) + C(s) \leq U(s) - C(s, t]$. Therefore

$$\begin{aligned} U(t) &\leq U(\sigma_U(t)) \vee \sup_{\sigma_U(t) \leq s \leq t} \{U(s) - C(s, t]\} \\ &= U(\sigma_U(t)) = \Phi(U)(t). \end{aligned}$$

Thus, $U \leq \Phi(U)$. On the other hand, since $U \in \mathcal{R}(X)$, we have $\Phi(U) \leq U$, by (12). \square

Lemma 9. *Let $B \in \mathcal{R}(X)$ be any fixed point of Φ . Then $B \leq U$.*

Proof. Since $B = \Phi(B) = B \circ \sigma_B$ we have

$$B = B \circ \sigma_B^{(k)}$$

where $\sigma_B^{(k)} := \underbrace{\sigma_B \circ \cdots \circ \sigma_B}_{k \text{ times}}$. Since

$$t \geq \sigma_B(t) \geq \sigma_B \circ \sigma_B(t) \geq \cdots \geq \sigma_B^{(k)}(t),$$

we may define

$$\sigma_B^{(\infty)}(t) := \lim_{k \rightarrow \infty} \sigma_B^{(k)}(t).$$

By the continuity of B ,

$$B = B \circ \sigma_B^{(\infty)}. \quad (17)$$

On the other hand, (11) gives

$$A \circ \sigma_B^{(k+1)} + B \circ \sigma_B^{(k+1)} = C \circ \sigma_B^{(k)}, \quad k \geq 1.$$

Taking the limit as $k \rightarrow \infty$, and using the continuity of A , B and C , we have

$$A \circ \sigma_B^{(\infty)} + B \circ \sigma_B^{(\infty)} = C \circ \sigma_B^{(\infty)}.$$

Since $A(t) + U(t) \geq C(t)$ for all t , we have

$$A \circ \sigma_B^{(\infty)} + U \circ \sigma_B^{(\infty)} \geq C \circ \sigma_B^{(\infty)},$$

and from the last two displays we conclude that

$$U \circ \sigma_B^{(\infty)} \geq B \circ \sigma_B^{(\infty)}.$$

Since U is nondecreasing and since (17) holds, we have

$$U \geq U \circ \sigma_B^{(\infty)} \geq B \circ \sigma_B^{(\infty)} = B,$$

as claimed. □

We are now ready to prove Theorem 1. We already know from Lemma 4 that $Q^* \leq Q^\infty$. So we only have to prove the opposite inequality. Recall that $Q_1 = A$ and $B_1 = C$. Trivially then

$$Q_1(t) + C(t) = A(t) + B_1(t), \quad t \geq 0.$$

Thus, for $0 \leq s \leq t$ we have

$$\begin{aligned} Q_1(s) > C(s, t] &\iff Q_1(s) + C(s) > C(t) \\ &\iff A(s) + B_1(s) > C(t) \\ &\iff s > \sigma_{B_1}(t). \end{aligned}$$

From this we get

$$\begin{aligned} Q_2(t) &= \int_0^t \mathbf{1}(Q_1(s) > C(s, t]) \, dA(s) \\ &= \int_0^t \mathbf{1}(s > \sigma_{B_1}(t)) \, dA(s) \\ &= A(t) - A(\sigma_{B_1}(t)). \end{aligned}$$

But (11) gives

$$A(\sigma_{B_1}(t)) + B_1(\sigma_{B_1}(t)) = C(t),$$

and so

$$Q_2(t) + C(t) = A(t) + B_1(\sigma_{B_1}(t)) = A(t) + B_2(t), \quad t \geq 0.$$

We now claim that

$$Q_k(t) + C(t) = A(t) + B_k(t), \quad t \geq 0, \quad k \geq 1.$$

This can be proved by induction along the same lines as above. Taking limits as $k \rightarrow \infty$, we conclude

$$Q_\infty(t) + C(t) = A(t) + B_\infty(t), \quad t \geq 0.$$

Lemma 7 tells us that B_∞ is a fixed point of Φ , and so, by Lemma 9,

$$B_\infty \leq U.$$

Hence

$$\begin{aligned} Q_\infty(t) + C(t) &= A(t) + B_\infty(t) \\ &\leq A(t) + U(t) \\ &= Q^*(t) + C(t), \quad t \geq 0, \end{aligned}$$

and this gives

$$Q_\infty \leq Q^*,$$

as needed. □

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